

Small Random Perturbation of Multi-dimensional Ordinary Differential Equation

Liangquan Zhang^{1,2 *}

1. School of Mathematics, Shandong University
Jinan 250100, People's Republic of China.

2. Laboratoire de Mathématiques,
Université de Bretagne Occidentale, 29285 Brest Cédex, France.

14/02/2012

Abstract

When the ordinary differential equation (ODE in short)

$$\begin{cases} \xi'(t) = b(\xi(t)), \\ \xi(0) = x \in \mathbf{R}^n, \end{cases}$$

where $b : \mathbf{R}^n \rightarrow \mathbf{R}^n$, has not a Lipschitz right hand side, there is neither existence nor uniqueness property of the associated Cauchy problem. However, the perturbed stochastic differential equation (SDE in short)

$$\begin{cases} dX^\varepsilon(t) = b(X^\varepsilon(t))dt + \varepsilon dW_t, \\ X^\varepsilon(0) = x \in \mathbf{R}^n, \end{cases}$$

where W is a n -dimensional standard Brownian motion, has a unique strong solution when b is only continuous and bounded. Moreover, when $\varepsilon \rightarrow 0$, the solutions to the perturbed SDE converges, in a sense, to the solutions of ODE. This phenomenon has been extensively studied in the literature in the one dimensional case. The goal of present paper is to analyzes the multi-dimensional case (this needs slightly different technique that in dimension one). In the case where the ODE has infinity many solutions, one of the main outcome of our approach is to explain which solutions of the ODE can be the limits of the solutions of the perturbed SDE.

*This work was partially supported by Marie Curie Initial Training Network (ITN) project: "Deterministic and Stochastic Controlled System and Application", FP7-PEOPLE-2007-1-1-ITN, No. 213841-2 and National Natural Science Foundation of China Grant 10771122, Natural Science Foundation of Shandong Province of China Grant Y2006A08 and National Basic Research Program of China (973 Program, No. 2007CB814900). Corresponding author, E-mail: xiaoquan51011@163.com.

Key words: Hamilton-Jacobi-Bellman equation (H-J-B equation), Ordinary differential equation (ODE in short), Stochastic differential equation (SDE in short), Small random perturbation.

A.M.S. Subject Classification: 34D10, 49L25, 46E35.

1 Introduction

Recall that the following ordinary differential equation (ODE in short)

$$\begin{cases} d\xi(t) = b(\xi(t)) dt, \\ \xi(0) = x, \quad t \geq 0, \end{cases} \quad (1.1)$$

may have many solutions or may have no solution at all if $b : \mathbf{R}^n \rightarrow \mathbf{R}^n$ is not Lipschitz continuous. However, one can regularize this equation by adding the white noise εdW_t to its right-hand side with any positive small intensity $\varepsilon > 0$ and n -dimensional Brownian motion W which is the n -dimensional coordinate process on the classical Wiener space (Ω, \mathcal{F}, P) , i.e., Ω is the set of continuous functions from $[0, +\infty)$ to \mathbf{R}^n starting from 0 ($\Omega = C([0, +\infty); \mathbf{R}^n)$), \mathcal{F} the completed Borel σ -algebra over Ω , P the Wiener measure and W the canonical process: $W_s(\omega) = \omega_s$, $s \in [0, +\infty)$, $\omega \in \Omega$. By $\{\mathcal{F}_s, 0 \leq s < +\infty\}$ we denote the natural filtration generated by $\{W_s\}_{0 \leq s < +\infty}$ and augmented by all P -null sets.

In other words, for any bounded Borel function $b : \mathbf{R}^n \rightarrow \mathbf{R}^n$, $x \in \mathbf{R}^n$ and n -dimensional Brownian motion W , there exists a unique strong solution of the following stochastic differential equation (SDE in short)

$$\begin{cases} dX^{x,\varepsilon}(t) = b(X^{x,\varepsilon}(t)) dt + \varepsilon dW_t, \\ X^\varepsilon(0) = x, \end{cases} \quad (1.2)$$

for any $\varepsilon > 0$. This result can be seen in [13], [15].

A natural question concerns the behavior of the limit of perturbed SDE (1.2) with respect to the ODE (1.1), as $\varepsilon \rightarrow 0$. In the classical Lipschitz case we refer to Friedlin and Wentzell [7]. In the case of where b is only continuous, more complex situation may occur. To show this phenomenon, we will explain precisely an easy one-dimensional example (taken from [4]) as follows:

Example 1. Put $b(x) = 2\text{sign}(x)\sqrt{|x|}$. The ODE

$$\begin{cases} \xi'(t) = 2\text{sign}(\xi(t))\sqrt{|\xi(t)|} \\ \xi(0) = 0. \end{cases} \quad (1.3)$$

has infinity many solutions that among them only both extremal solutions " $t \rightarrow t^2$ and $t \rightarrow -t^2$ " are limiting of the corresponding SDE. More precisely, the limit of (1.2) are continuous stochastic processes denoted by $X(\cdot)$ which satisfies

$$\begin{cases} P(X(t) = t^2) = \frac{1}{2}, \\ P(X(t) = -t^2) = \frac{1}{2}. \end{cases}$$

In the literature, the general one-dimensional case was for instance considered by Bafico, et al. [3, 4]. They showed that the limit of perturbed SDE (1.2) are processes which support belong to the solution of ODE (1.1). In Gradinaru et al. [11], using the large deviation technique a more precise description of the limit provided in the case

$$\xi'(t) = 2\text{sign}(\xi(t)) |\xi(t)|^\gamma, \quad \gamma \in (0, 1).$$

For the multi-dimensional case the outcome of Buckdahn et al. [6] shows that the limit has support in the set of solutions of ODE (1.1). Our main aim is to provide a more precise description to the limit (consider the Example 1, the constant solution $\xi(t) \equiv 0$ is a solution of ODE (1.3) but is not in the support of the limit processes). For other discussion, see [9].

To our best knowledge, the existing technique of one-dimensional case is based on the explicit computation of the laws of $X^\varepsilon(\cdot)$. Indeed, in [3], [4], the laws are allowed by studying the Boundary Value Problems. In [10], laws are derived from the large deviation theory as analytic tools. However, in the multi-dimensional case, many computations of laws is hardly possible.

Our approach to the problem (1.1), (1.2) is different from the above mentioned ones, and it is based on the use of exit time and related first and second order Hamilton-Jacobi-Bellman equation (H-J-B equation). We shall show that the exit time of SDE (1.2) as the solution of the second order H-J-B equation converges to exit time of optimal solution (defined in next section) for ODE (1.1), as $\varepsilon \rightarrow 0$. Then, we are able to show (under certain conditions on b) that the optimal solutions of ODE (1.1) are cluster point, with some positive probability.

This paper is organized as follows: In the next section we present some preliminaries. Section 3 is concern with the properties of exit time for deterministic systems, especially, the continuity of exit time function. Section 4 is devoted to study the first and second order H-J-B equation arising from the exit time problems. After having characterized the value functions $\mathcal{V}_K(\mathcal{U}^\varepsilon)$ defined below, as a unique continuous viscosity solutions (unique Sobolev and also continuous solution) of associated first (second) order H-J-B equation, we give the main result in Section 5 with some examples.

2 Preliminary

Next we define the exit time of a continuous function $x(\cdot) \in C([0, T]; \mathbf{R}^n)$. Some notations will be given in following sections if necessary.

Definition 2. Let $K \subset \mathbf{R}^n$ be a closed subset and $x(\cdot) \in C([0, +\infty); \mathbf{R}^n)$ be a continuous function. We denote by

$$\tau_K : C([0, +\infty); \mathbf{R}^n) \rightarrow R_+ \cup \{+\infty\}$$

the exit functional associating with $x(\cdot)$ its exit time $\tau_K(x(\cdot))$ defined by

$$\tau_K(x(\cdot)) \doteq \inf \{t \in [0, +\infty) | x(t) \notin K\}.$$

We observe that

$$\forall t \in [0, \tau_{\mathcal{K}}(x(\cdot))), \quad x(t) \in \mathcal{K},$$

and that, when $\tau_{\mathcal{K}}(x(\cdot))$ is finite,

$$x(\tau_{\mathcal{K}}(x(\cdot))) \in \partial\mathcal{K}.$$

We use the convention $\inf \{\phi\} = +\infty$, so that $x(\tau_{\mathcal{K}}(x(\cdot)))$ is infinite means that $x(t) \in \mathcal{K}$ for all $t \geq 0$.

We will need the following lemma (see in [1] Lemma 4.2.2. page 134).

Lemma 3. *Let $\mathcal{K} \subset \mathbf{R}^n$ be a closed subset. The functional $\tau_{\mathcal{K}}$ is an upper semicontinuous when $C([0, +\infty); \mathbf{R}^n)$ is supplied with the pointwise convergence topology.*

Consider the following assumption

(H1): $b(\cdot) : \mathbf{R}^n \rightarrow \mathbf{R}^n$, is bounded and continuous.

Recall that, under the assumption (H1), SDE (1.2) has a unique strong solution (for more information see [15]).

3 Ordinary Differential Equation

In general, under the assumptions (H1), the uniqueness of solution for (1.1) may fail. Hence, we denote $\mathcal{S}_x(b)$ the trajectories of solutions of (1.1) starting from x if (1.1) admits many solutions. Denote $\mathcal{K} = \overline{\text{Int}(\mathcal{K})}$ a compact subset of \mathbf{R}^n such that $0 \in \text{Int}(\mathcal{K})$. Typically, $\mathcal{K} = \overline{B(0, 1)}$.

Set $x = 0$ and note that if $b(0) \neq 0$, the trajectories of solutions will leave the center point, but if $b(0) = 0$, the equation (1.1) may have zero solution which means that the trajectory will never touch the boundary of the closed ball \mathcal{K} . Replacing b by $b(x) - b(0)$ if necessary, we then attain a new ODE with $\xi(t) \equiv 0$ is a solution. For simplicity, we assume the following assumption.

(H2): $b(0) = 0$ and $0 \notin \text{Int}\{x | b(x) = 0\}$.

Indeed, clearly if $0 \in \text{Int}\{x | b(x) = 0\}$ the constant solution is the only solution to (1.1) with $\xi(t) = 0$. Actually, (H2) means that not only the zero is one of the solutions of (1.1), but also there exist many other non-zero solutions.

From now on, we are concerned with the solutions of (1.1) which reach the boundary of \mathcal{K} as fast as possible. Let us denote the exit time function as follows:

$$\mathcal{V}_{\mathcal{K}}(x) = \inf_{\xi^x(\cdot) \in \mathcal{S}_x(b)} [\tau_{\mathcal{K}}(\xi^x(\cdot))], \quad x \in \mathcal{K}, \quad (3.3)$$

where \mathcal{K} is a closed and bounded subset of \mathbf{R}^n .

We give the following.

Lemma 4. Assume (H1) and (H2) hold. If $\mathcal{V}_\mathcal{K}(0) < +\infty$, for some suitable \mathcal{K} , then there exist optimal solutions $\bar{\xi}^0(\cdot) \in \mathcal{S}_0(b)$ of (1.1), such that

$$\bar{\xi}^0(\mathcal{V}_\mathcal{K}(0)) \in \partial\mathcal{K} \text{ and } \bar{\xi}^0((0, \mathcal{V}_\mathcal{K}(0))) \in \text{Int}(\mathcal{K}). \quad (3.4)$$

The proof is not difficult, we omit it.

Remark 5. In [3], the author also gave the similar assumption, that is, for some $r > 0$ the exit time $\mathcal{V}_\mathcal{K}$ is finite. Actually, for one-dimensional case, the expression of exit time can be obtained easily. But for general case, it is impossible to give the explicit expression.

Remark 6. It is necessary to point out that the trajectories of the optimal solution may not be unique (see Example 1).

On the other hand, in order to prove that $\mathcal{V}_\mathcal{K}$ is a lower semicontinuous function, we give the following assumption. The idea is borrowed from [12].

(H3): Assume that

$$\langle b(x), \vec{n}(x) \rangle > 0, \quad \forall x \in \partial\mathcal{K},$$

where $\vec{n}(x)$ denotes the unit outward normal. Furthermore, we assume $\partial\mathcal{K}$ is C^1 , so that $\vec{n}(x)$ is continuous and well-defined.

Clearly, (H3) means that as soon as of every trajectory of (1.1) reaches the boundary of \mathcal{K} , then it leaves the ball immediately. However, as we have shown in Lemma 5, there may be infinite many optimal solutions starting from 0 which means that it is a singular point. Hence, we have the following.

Lemma 7. Under the assumptions (H1), (H2) and (H3), for some compact domain \mathcal{K} , the function $\mathcal{V}_\mathcal{K} : \mathcal{K}/\{0\} \rightarrow \mathbf{R}_+ \cup \{+\infty\}$ is a continuous function.

Proof. Firstly, the function $\mathcal{V}_\mathcal{K}$ being on infimum of a family of upper semicontinuous function (in view of formula (3.3) and Lemma 3), consequently, it is upper semicontinuous. It is remain to show that $\mathcal{V}_\mathcal{K}$ is a lower semicontinuous function. Fix $x \in \mathcal{K}/\{0\}$. We wish to prove that $\mathcal{V}_\mathcal{K}$ is lower semicontinuous, namely,

$$\liminf_{y \rightarrow x} \mathcal{V}_\mathcal{K}(y) \geq \mathcal{V}_\mathcal{K}(x). \quad (3.6)$$

This is clearly valid if $\mathcal{V}_\mathcal{K}(x) = 0$ on $\liminf_{y \rightarrow x} \mathcal{V}_\mathcal{K}(y) < +\infty$. Assume that $\mathcal{V}_\mathcal{K}(x) > 0$ and $\liminf_{y \rightarrow x} \mathcal{V}_\mathcal{K}(y) < +\infty$.

Suppose, on the contrary, that (3.6) does not hold true. Hence there exists $\varepsilon > 0$, and $x_n \in \mathcal{K}$ with $|x_n - x| < \frac{1}{n}$ and

$$\bar{t} := \lim_{n \rightarrow \infty} \mathcal{V}_\mathcal{K}(x_n) \leq \mathcal{V}_\mathcal{K}(x) - \varepsilon.$$

By lemma 5, there exist $\bar{\xi}_n^{x_n}(t) \in \mathcal{S}_{x_n}(b)$ such that $\mathcal{V}_K(x_n) = \tau_K(\bar{\xi}_n^{x_n}(\cdot))$ and

$$\bar{\xi}_n^{x_n}([0, \mathcal{V}_K(x_n)]) \subset \mathcal{K}. \quad (3.7)$$

Because b is bounded, then it is easy to check that $(\bar{\xi}_n^{x_n}(\cdot))_{n \geq 1}$ are equicontinuous. Hence, using Arzela-Ascoli Theorem, we know that there exist $\bar{\xi}_n^{x_n}(\cdot) \rightarrow \bar{\xi}^x(\cdot)$ uniformly on the $[0, T]$ (up to a subsequence).

But (H3) implies that $\bar{\xi}^x(\cdot)$ leaves \mathcal{K} instantly after \bar{t} , and from (3.7) we have $\bar{\xi}^x([0, \bar{t}]) \subset \mathcal{K}$. Thus we obtain $\tau_K(\bar{\xi}^x(\cdot)) = \bar{t}$, which is contradiction to $\bar{t} \leq \mathcal{V}_K(x) - \varepsilon$. The proof is complete. \square

Now let us recall the following result adapted to Theorem 4 in [6].

Proposition 8. *Suppose that (H1) holds. Let $X^{x, \varepsilon}$ be a strong solution to the SDE [1.2]. Then, there exists $\varepsilon_n \rightarrow 0$ such that $X^{x, \varepsilon_n}(\cdot)$ converges in law, as $\varepsilon_n \rightarrow 0$, to some $X^0(\cdot)$ which belongs almost surely to the set of solution to (1.1).*

The proof being similar to Theorem 4 in [6] we omit it. Hence, a natural question is that what the solutions belonging to set of solutions of (1.1) are the limit solutions of (1.2) with some positive probability.

4 First and Second order H-J-B equation

In this subsection, we are devoted to investigate the first and second order H-J-B equation respectively arising from the exit problem. For the first order one, because the continuity of \mathcal{V}_K has been already proved in Section 3 on some special domain. Therefore, viscosity solution theory may be applied. While for second order one, we use the theory of elliptic PDEs of second order.

4.1 First order H-J-B equation

Consider the following first order H-J-B equation

$$\begin{cases} -\langle b(x), Du(x) \rangle - 1 = 0, & x \in \mathcal{D}, \\ u(x) = 0, & x \in \partial\mathcal{D}, \end{cases} \quad (4.1.1)$$

where b satisfies (H1) and \mathcal{D} is a bounded open subset of \mathbf{R}^n .

We want to prove that the exit time as a value function without control for deterministic system is the unique viscosity solution of equation (4.1.1). Let us first recall the definition of a viscosity solution of equation (4.1.1).

Definition 9. *A real-valued continuous function $u \in C(\mathbf{R}^n; \mathbf{R})$ is called*

(i) a viscosity subsolution of equation (4.1.1) if $u(x) \leq 0$, for all $x \in \partial\mathcal{D}$, and if for all functions $\varphi \in C^1(\mathbf{R}^n; \mathbf{R})$ and $x \in \mathcal{D}$ such that $u - \varphi$ attains its local maximum at x

$$-\langle b(x), D\varphi(x) \rangle - 1 \leq 0.$$

(ii) a viscosity supersolution of equation (4.1.1) if $u(x) \geq 0$, for all $x \in \partial\mathcal{D}$, and if for all functions $\varphi \in C^1(\mathbf{R}^n; \mathbf{R})$ and $x \in \mathcal{D}$ such that $u - \varphi$ attains its local minimum at x

$$-\langle b(x), D\varphi(x) \rangle - 1 \geq 0.$$

(iii) a viscosity solution of equation (4.1.1) if it is both a viscosity sub- and a supersolution of equation (4.1.1).

We show first that the value function $\mathcal{V}_{\mathcal{K}}(x)$ is a continuous viscosity solution of equation (4.1.1). The proof is classical but it is written for sake of completeness.

Proposition 10. *Under the assumptions in Lemma 6, we claim that $\mathcal{V}_{\mathcal{K}}$ is a continuous viscosity solution of the following first order H-J-B equation*

$$\begin{cases} \sum_{i=1}^n \frac{\partial u(x)}{\partial x_i} b_i(x) = -1, & x \in \mathcal{K}, \\ u(x) = 0, & x \in \partial\mathcal{K}. \end{cases} \quad (4.1.2)$$

Proof. Obviously, $\mathcal{V}_{\mathcal{K}}(x) = 0$, $x \in \partial\mathcal{K}$. Let us show in a first step that $\mathcal{V}_{\mathcal{K}}$ is a viscosity super-solution. For this we suppose that, for all $\varphi \in C^1(\mathbf{R}^n; \mathbf{R})$, whenever $x_0 \in \mathcal{K}$ is a point of local minimum of $\mathcal{V}_{\mathcal{K}} - \varphi$, that is

$$\begin{cases} \varphi(x_0) = \mathcal{V}_{\mathcal{K}}(x_0) \\ \varphi(y) \leq \mathcal{V}_{\mathcal{K}}(y) \quad y \neq x_0. \end{cases}$$

Then combining to the definition of $\mathcal{V}_{\mathcal{K}}$ and the optimal solution $\bar{\xi}^{x_0}(\cdot)$ obtained in Lemma 5, we have

$$\varphi(x_0) = \mathcal{V}_{\mathcal{K}}(\bar{\xi}^{x_0}(t)) + t \geq \varphi(\bar{\xi}^{x_0}(t)) + t,$$

that is

$$\frac{\varphi(\bar{\xi}^{x_0}(t)) - \varphi(x_0)}{t} \leq -1,$$

Letting t tend to 0, we get

$$\langle D\varphi(x_0), b(x_0) \rangle \leq -1.$$

Hence $\mathcal{V}_{\mathcal{K}}$ is a viscosity super-solution. The proof to sub-solution being similar we omit it. The proof is complete. \square

Now set

$$H(x, p) = -\langle b(x), p \rangle - 1, \quad \forall x \in \mathbf{R}^n, \quad \forall p \in \mathbf{R}^n. \quad (4.1.3)$$

We recall the following a comparison principle for (4.1.1) from

Proposition 11. (Theorem 5.9, Page 82, in [5]) Let \mathcal{D} be a bounded open subset of \mathbf{R}^n . Assume $u_1, u_2 \in C(\overline{\mathcal{D}})$ are, respectively viscosity sub- and super-solution of (4.1.1) in \mathcal{D} with $u_1 \leq u_2$ on $\partial\mathcal{D}$. Assume that H satisfies

$$|H(x, p) - H(y, p)| \leq w_1(|x - y|(1 + |p|)) \quad (4.1.4)$$

where w_1 is the modulus¹ of H , for $x, y \in \mathcal{D}$ and $p \in \mathbf{R}^n$. Also,

$$p \rightarrow H(x, p) \text{ is convex on } \mathbf{R}^n \text{ for each } x \in \mathcal{D}; \quad (4.1.5)$$

there exists

$$\varphi \in C(\overline{\mathcal{D}}) \cap C^1(\mathcal{D}) \quad (4.1.6)$$

such that $\varphi \leq u_1$ in $\overline{\mathcal{D}}$ and

$$\sup_{x \in \mathcal{D}'} H(x, D\varphi(x)) < 0, \quad \forall \overline{\mathcal{D}'} \subseteq \mathcal{D}. \quad (4.1.7)$$

Then

$$u_1 \leq u_2, \text{ in } \mathcal{D}.$$

As a straightforward consequence we attain the following.

Lemma 12. Under the assumptions Proposition 9, we claim that $\mathcal{V}_{\mathcal{K}}$ is a unique continuous viscosity solution to the H-J-B equation (4.1.2).

Proof. Clearly, it is easy to check that $H(x, p) = -\langle b(x), p \rangle - 1$ satisfies (4.1.4) and (4.1.5). Indeed, we have

$$\begin{aligned} |H(x, p) - H(y, p)| &= |\langle b(x) - b(y), p \rangle| \\ &\leq w_b(|x - y|) |p| \\ &\leq w_b(|x - y|) (1 + |p|). \end{aligned}$$

The second inequality holds by Schwartz's inequality. To verify (4.1.7), setting $\varphi(x) = \min_{y \in \mathcal{K}} \mathcal{V}_{\mathcal{K}}(y)$, $x \in \mathcal{K}$, by Proposition 10, we get the desired result. \square

¹We say that b is a uniformly continuous function, that is, there exists a $w_b(r)$ modulus of continuity (i.e. $w_b : [0, +\infty) \rightarrow [0, +\infty)$ continuous, nondecreasing with $w_b(0) = 0$) such that

$$|b(x) - b(y)| \leq w_b(|x - y|), \quad \forall x, y \in \text{domain of } b.$$

4.2 Second order H-J-B Equation

Consider the SDE (1.1) and define

$$\theta^\varepsilon(x) \doteq \inf \{t \geq 0 : X^{x,\varepsilon}(t) \notin \mathcal{K}\}, \quad \forall x \in \mathcal{K}, \varepsilon > 0,$$

and

$$\mathcal{U}^\varepsilon(x) = \mathbf{E}[\theta^\varepsilon(x)], \quad x \in \mathcal{K}, \varepsilon > 0.$$

We introduce some notations. For any $q > 1$, denote by $W^{2,q}(\mathcal{K})$ the Sobolev space equipped with the following norm:

$$\|u\|_{W^{2,q}(\mathcal{K})} := \|u\|_{L^q(\mathcal{K})} + \|\partial_x u\|_{L^q(\mathcal{K})} + \|\partial_x^2 u\|_{L^q(\mathcal{K})},$$

where

$$\|f\|_{L^q(\mathcal{K})} := \left(\int_{\mathcal{K}} |f(x)|^q dx \right)^{\frac{1}{q}}.$$

From now on, we assume that \mathcal{K} is a compact manifold² of class $(C^{1,1})^3$. Let us recall the following lemmas.

Lemma 13. *(Theorem 9.15 and Corollary 9.18, Page 243, in [10]) Assume that (H1) holds. Then, for fixed $\varepsilon > 0$, the following elliptic-type PDE*

$$\begin{cases} \mathcal{L}^\varepsilon u^\varepsilon(x) = 1, & x \in \mathcal{K}, \\ u^\varepsilon(x) = 0, & x \in \partial\mathcal{K}, \end{cases} \quad (4.2.1)$$

where

$$\mathcal{L}^\varepsilon = \sum_{i=1}^n b(x) \frac{\partial}{\partial x_i} + \frac{\varepsilon^2}{2} \sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j},$$

²Assume that K be a compact C^1 manifold in \mathbf{R}^n , hence there exists a C^1 function $\psi : \mathbf{R}^n \rightarrow \mathbf{R}$, such that

$$K = \{x \in \mathbf{R}^n : \psi(x) \leq 0\}.$$

and

$$\text{for } \forall x \in \mathbf{R}^n, \psi(x) = 0 \Rightarrow \nabla \psi(x) \neq 0.$$

Consequently, we have

$$\begin{cases} \overset{\circ}{K} = \{x \in \mathbf{R}^n : \psi(x) < 0\} \neq \emptyset, \\ \partial K = \{x \in \mathbf{R}^n : \psi(x) = 0\} \neq \emptyset. \end{cases}$$

For instance when

$$\psi(x) = |x|^2 - r, \text{ for some } r > 0,$$

the associated set K is the closed ball

$$\mathcal{K}_r = \{x \in \mathbf{R}^n : \psi(x) \leq 0\}.$$

³Here $C^{1,1}$ denotes the Hölder space consisting of functions whose 1-st order partial derivatives are uniformly Hölder continuous with exponent $\alpha = 1$.

has a unique solution $u^\varepsilon \in (W_0^{1,p}(\mathcal{K}))^4 \cap W^{2,p}(\mathcal{K}) \cap C(\mathcal{K})$, for some $p > \frac{n}{2}$.

We have the following Krylov's estimate.

Lemma 14. *Assume that (H1) holds. Then for any Borel function $f(x) \in \|f\|_{L^q(\mathcal{K})}$, and $q > n + 2$, we have*

$$\mathbf{E} \left[\int_0^{T \wedge \theta^\varepsilon(x)} f(X^{x,\varepsilon}(t)) dt \right] \leq N \|f\|_{L^q(\mathcal{K})}, \quad (4.2.2)$$

where N is a constant depending only on \mathcal{K} , ε , T and b .

We omit the proof of Lemma 13 since it is very similar to that of Theorem 3 in [14]. The following result generalizes Krylov's extension of Itô's formula.

Lemma 15. *Assume that (H1) holds. Then for any $u : \mathbf{R}^n \rightarrow \mathbf{R}$ from the Sobolev space $W^{2,p}(\mathcal{K})$, $p > n + 2$ we have*

$$u(X^{x,\varepsilon}(t)) - u^\varepsilon(x) = \int_0^t (\mathcal{L}^\varepsilon u(X^{x,\varepsilon}(s))) ds + \varepsilon \int_0^t \partial_x u(X^{x,\varepsilon}(s)) dW_s, \quad (4.2.3)$$

almost surely for $t \leq \theta^\varepsilon(x) \wedge T$, where $T > 0$.

Proof. The proof is classical. For the convenience of the reader we give the details of the proof. At the beginning, we show that each integral in (4.2.3) is well-defined. Noting that \mathcal{K} is compact manifold of class $C^{1,1}$, by virtue of Sobolev's embedding theorem there exists a constant N such that

$$\sup_{x \in \mathcal{K}} \left(|u(x)| + \sum_i |\partial_{x_i} u(x)| \right) \leq N \|u\|_{W^{2,p}(\mathcal{K})}.$$

for all $u \in W^{2,p}(\mathcal{K})$, $p > n + 2$. Hence

$$\varepsilon^2 \mathbf{E} \left[\int_0^{\theta^\varepsilon(x) \wedge T} |\partial_x u(X^{x,\varepsilon}(s))|^2 ds \right] \leq \varepsilon^2 T \|u\|_{W^{2,p}(\mathcal{K})}^2, \quad (4.2.4)$$

and

$$\mathbf{E} \left[\int_0^{\theta^\varepsilon(x) \wedge T} |b(X^{x,\varepsilon}(s)) \partial_{x_i} u(x)| ds \right] \leq MT \|u\|_{W^{2,p}(\mathcal{K})}^2, \quad (4.2.5)$$

by the boundness of b . Moreover,

$$\varepsilon^2 \mathbf{E} \left[\int_0^{\theta^\varepsilon(x) \wedge T} |\partial_{x_i, x_j} u(X^{x,\varepsilon}(s))| ds \right] \leq \varepsilon^2 N \|\partial_{x_i, x_j} u\|_{L^p(\mathcal{K})} \leq \varepsilon^2 NT \|u\|_{W^{2,p}(\mathcal{K})}, \quad (4.2.6)$$

⁴ $W_0^{1,p}(\mathcal{K})$ arises by taking the closure of $C_0^1(\mathcal{K})$ in $W^{1,p}(\mathcal{K})$ where $C_0^1(\mathcal{K})$ denotes the set of functions in $C^1(\mathcal{K})$ with compact support in \mathcal{K} .

by Lemma 13. Consequently, the right-hand side of (4.2.3) is well-defined for $t \leq T \wedge \theta^\varepsilon(x)$.

Actually, for $u \in W^{2,p}(\mathcal{K})$ there exists a sequence of function u_n in $C^2(\mathbf{R}^n)$ such that

$$\|u_n - u\|_{W^{2,p}(\mathcal{K})} \rightarrow 0, \quad \text{as } n \rightarrow +\infty.$$

Applying Itô's formula we have

$$\begin{aligned} & u_n(X^{x,\varepsilon}(t \wedge \theta^\varepsilon(x) \wedge T)) - u_n(x) \\ &= \int_0^{t \wedge \theta^\varepsilon(x) \wedge T} (\mathcal{L}^\varepsilon u_n(X^{x,\varepsilon}(s))) \, ds + \varepsilon \int_0^{t \wedge \theta^\varepsilon(x) \wedge T} \partial_x u_n(X^{x,\varepsilon}(s)) \, dW_s. \end{aligned} \tag{4.2.7}$$

for any $t \geq 0$. On the other hand, the inequalities (4.2.4), (4.2.5), (4.2.6) hold with $u - u_n$ replacing of u , with constants independent of n . Lastly, letting $n \rightarrow +\infty$ in (4.2.7) we get the desired result. \square

We are now in a position to make a important observations as follows:

Lemma 16. *Let $u^\varepsilon \in W_0^{1,p}(\mathcal{K}) \cap W^{2,p}(\mathcal{K}) \cap C(\mathcal{K})$, $p > n+2$, be a unique solution of (4.2.1). Under the assumption (H1), we have*

$$u^\varepsilon(x) = \mathcal{U}^\varepsilon(x), \quad x \in \mathcal{K}.$$

Proof. Applying Lemma 14 to u^ε and taking the expectation, we have

$$\begin{aligned} & \mathbf{E}[u^\varepsilon(X^{x,\varepsilon}(\theta^\varepsilon(x))) - u^\varepsilon(x)] \\ &= \mathbf{E}\left[\int_0^{\theta^\varepsilon(x)} (\mathcal{L}^\varepsilon u^\varepsilon(X^{x,\varepsilon}(s)) + 1) \, ds - \int_0^{\theta^\varepsilon(x)} 1 \, ds + \varepsilon \int_0^{\theta^\varepsilon(x)} \partial_x u^\varepsilon(X^{x,\varepsilon}(s)) \, dW_s\right]. \end{aligned}$$

Since

$$\begin{cases} \mathbf{E}[u^\varepsilon(X^{x,\varepsilon}(\theta^\varepsilon(x)))] = 0, \\ \mathbf{E}[\mathcal{L}^\varepsilon u^\varepsilon(X^{x,\varepsilon}(s)) + 1] = 0, \\ \mathbf{E}\left[\varepsilon \int_0^{\theta^\varepsilon(x)} \partial_x u^\varepsilon(X^{x,\varepsilon}(s)) \, dW_s\right] = 0, \end{cases}$$

we obtain

$$u^\varepsilon(x) = \mathbf{E}[\theta^\varepsilon(x)] = \mathcal{U}^\varepsilon(x), \quad x \in \mathcal{K}.$$

The proof is complete. \square

Remark 17. *Because the diffusion term is not degenerated, the θ^ε is almost surely finite.*

5 Main Results

In Section 4, we have studied the first and second order H-J-B equation, respectively. However, a question is that what is the relationship between $\mathcal{V}_\mathcal{K}$ and \mathcal{U}^ε as $\varepsilon \rightarrow 0$. To answer it, we first give the following Lemma 16. Then, we shall be devoted to show our main result.

Lemma 18. *Assume that b satisfies the assumptions (H1) and (H2) hold. Let \mathcal{K} be a suitable compact manifold of class C^1 , such that (H3) holds. Then for almost everywhere $\lim_{\varepsilon \rightarrow 0} \theta^\varepsilon(x)$ exists, $x \in \mathcal{K}$. Furthermore we have,*

$$\lim_{\varepsilon \rightarrow 0} \theta^\varepsilon(x) = \theta(x), \text{ } P\text{-a.s.} \quad (5.1)$$

where θ is the exit time of limiting solution of SDE (1.2).

Proof. Considering the perturbed SDE (1.2), one can easily show that the family of laws

$$\left\{ P \circ (X^{\varepsilon,0})^{-1}, \varepsilon > 0 \right\}$$

is tight. Hence, by virtue of Prokhorov's theorem, there exists a sequence $\varepsilon_n \rightarrow 0^+$ with

$$\left\{ P \circ (X^{\varepsilon_n,0})^{-1} \right\} \rightarrow \left\{ P \circ (X^{\varepsilon,0})^{-1} \right\}, \text{ as } n \rightarrow +\infty.$$

Then using the Skohorod's theorem we can find a new probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ and stochastic processes $\tilde{X}^{\varepsilon_n}, \tilde{X}$ defined on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$, such that

$$\begin{cases} (1) \tilde{P} \circ (\tilde{X}^{\varepsilon_n}(\cdot))^{-1} = P \circ (X^{\varepsilon_n}(\cdot))^{-1} \text{ and } \tilde{P} \circ (\tilde{X}(\cdot))^{-1} = P \circ (X(\cdot))^{-1} \\ (2) \text{ in the topology of the uniform convergence on compact, } \tilde{X}^{\varepsilon_n}(\cdot) \rightarrow \tilde{X}(\cdot), \text{ a.s.} \end{cases} \quad (5.4)$$

Hence, for arbitrarily given $T > 0$,

$$\tilde{X}^{\varepsilon_n}(\cdot) \rightarrow \tilde{X}(\cdot) \text{ in } C([0, T]; \mathbf{R}^n), \quad \tilde{P}\text{-a.s.} \quad (5.5)$$

We now prove the lemma by contradiction. Set

$$A := \left\{ \omega \in \tilde{\Omega} \left| \limsup_{\varepsilon \rightarrow 0} \tilde{\theta}^\varepsilon(x, \omega) > \tilde{\theta}(x, \omega) \right. \right\}.$$

Suppose that $P(A) > 0$. Now consider the following limiting SDE association with stochastic system (1.2) as $\varepsilon \rightarrow 0$,

$$\tilde{X}(t) - \tilde{X}(0) = \int_0^t b(\tilde{X}(s)) \, ds, \quad \tilde{X}(0) \in \partial\mathcal{K}, \quad \forall t \geq 0.$$

Define a closed set $\mathcal{K}_\lambda := \{x \mid \psi(x) \leq \lambda, 0 < \lambda < 1\}$, where ψ is a C^1 function corresponding to \mathcal{K} . Obviously, $\mathcal{K} \subset \mathcal{K}_\lambda$. By assumption (H4), we define

$$2\alpha C_\lambda := \inf_{x \in \partial \mathcal{K}} \langle b(x), \nabla \psi(x) \rangle > 0,$$

where C_λ is the local Lipschitz constant of ψ on \mathcal{K}_λ . Taking $\eta \in (0, \lambda)$ small enough, such that

$$\forall y \in \{x : -\eta < \psi(x) < \eta\},$$

we have

$$\langle b(y), \nabla \psi(y) \rangle > C_\lambda \alpha$$

since the continuity of b . In particular, picking $\bar{t} > 0$ such that $\forall t \in [0, \bar{t}]$, we have

$$-\eta < \psi(\tilde{X}(t)) < \eta.$$

Note that

$$\frac{1}{2} \frac{d}{dt} \psi(\tilde{X}(t)) = \langle \nabla \psi(\tilde{X}(t)), \tilde{X}'(t) \rangle = \langle \nabla \psi(\tilde{X}(t)), b(\tilde{X}(t)) \rangle > \alpha C_\lambda.$$

Thus

$$\psi(\tilde{X}(t)) > 2\alpha C_\lambda t, \quad t \in [0, \bar{t}].$$

On the other hand, combining (5.4) and (5.5), we have

$$\left| X^{\varepsilon_n}(\theta(x, \omega) + \bar{t}) - \tilde{X}(\bar{t}) \right| < \alpha \bar{t}, \quad \text{for } n \text{ large enough.}$$

Note that

$$\tilde{X}(\bar{t}) = X(\theta(x, \omega) + \bar{t}) \text{ and } \psi \in C^1.$$

Consequently, for any $\omega \in A$, we have

$$\begin{aligned} \psi(X^{\varepsilon_n}(\theta(x, \omega) + \bar{t})) &\geq \psi(\tilde{X}(\bar{t})) - C_\lambda \left| \tilde{X}(\bar{t}) - X^{\varepsilon_n}(\theta(x, \omega) + \bar{t}) \right| \\ &\geq 2\alpha \bar{t} C_\lambda - \alpha \bar{t} C_\lambda > 0 \end{aligned}$$

which implies that

$$\tilde{\theta}^{\varepsilon_n}(x, \omega) \leq \tilde{\theta}(x, \omega) + \bar{t}.$$

Passing to limsup as $n \rightarrow +\infty$, we obtain

$$\limsup_{n \rightarrow +\infty} \tilde{\theta}^{\varepsilon_n}(x, \omega) \leq \tilde{\theta}(x, \omega),$$

which is contradiction with the definition of $\omega \in A$.

Now we define

$$B \doteq \left\{ \omega \in \tilde{\Omega} \mid \liminf_{\varepsilon \rightarrow 0} \tilde{\theta}^\varepsilon(x, \omega) < \tilde{\theta}(x, \omega) \right\}.$$

Fixing $\omega \in B$, for any $\delta \in [0, \theta(x, \omega))$, It is easy prove by absurdum that there exists $n > n(\delta)$, such that

$$\left| \tilde{X}^{\varepsilon_n} \left(\tilde{\theta}(x, \omega) - \delta \right) - \tilde{X} \left(\tilde{\theta}(x, \omega) - \delta \right) \right| + \psi \left(\tilde{X} \left(\tilde{\theta}(x, \omega) - \delta \right) \right) < 0,$$

from which we conclude that

$$\tilde{\theta}^{\varepsilon_n}(x, \omega) \geq \tilde{\theta}(x, \omega) - \delta.$$

Since arbitrary of δ , we have

$$\liminf_{n \rightarrow +\infty} \tilde{\theta}^{\varepsilon_n}(x, \omega) \geq \tilde{\theta}(x, \omega),$$

which is contradiction with the definition of $\omega \in B$.

Now we end the proof by observing that

$$1 = \tilde{P} \left[\lim_{\varepsilon \rightarrow 0} \tilde{\theta}^\varepsilon(x) = \tilde{\theta}(x) \right].$$

Because of (5.4), we can conclude that

$$1 = \tilde{P} \left[\lim_{\varepsilon \rightarrow 0} \tilde{\theta}^\varepsilon(x) = \tilde{\theta}(x) \right] = P \left[\lim_{\varepsilon \rightarrow 0} \theta^\varepsilon(x) = \theta(x) \right].$$

The proof is complete. \square

Now Let us introduce the following definitions of viscosity solution for second-order PDE:

$$F(x, V, DV, D^2V) = 0, \quad x \in \Omega, \quad (5.5)$$

where $\Omega \subset \mathbf{R}^n$ denotes an open set.

Definition 19. (i) We say that $V \in C(\Omega)$ is a viscosity subsolution of (5.5) at a point $x_0 \in \Omega$, if and only if, for any test function $\varphi \in C^2(\Omega)$ such that $V - \varphi$ has a local maximum at x_0 , then

$$F(x_0, V(x_0), D\varphi(x_0), D^2\varphi(x_0)) \leq 0; \quad (5.6)$$

(ii) We say that $V \in C(\Omega)$ is a viscosity supersolution of (5.5) at a point $x_0 \in \Omega$, if and only if, for any test function $\varphi \in C^2(\Omega)$ such that $V - \varphi$ has a local minimum at x_0 , then

$$F(x_0, V(x_0), D\varphi(x_0), D^2\varphi(x_0)) \geq 0; \quad (5.7)$$

(iii) We say that V is a viscosity solution in the open set Ω if V is a viscosity subsolution and a viscosity supersolution, at any point $x_0 \in \Omega$.

We will characterize the mean of exit time as unique viscosity solution of the H-J-B equation (4.2.1).

Lemma 20. Let $\varepsilon > 0$ be fixed. Then, \mathcal{U}^ε is a unique nonnegative viscosity solution of H-J-B equation (4.2.1).

Proof. The uniqueness is obtained from Lemma 12. We shall prove \mathcal{U}^ε is the viscosity solution of (4.1.2). Obviously, $\mathcal{U}^\varepsilon(x) = 0$, $x \in \partial\mathcal{K}$. Let us show in a first step that \mathcal{U}^ε is a viscosity super-solution. For this we suppose that, for all compactly-supported $C^2(\mathbf{R}^n; \mathbf{R})$ (twice differentiable with continuous second derivatives) function $\varphi : \mathbf{R}^n \rightarrow \mathbf{R}$, whenever x_0 is a point of local minimum of $\mathcal{U}^\varepsilon - \varphi$, that is

$$\begin{cases} \varphi(x_0) = \mathcal{U}^\varepsilon(x_0), \\ \varphi(y) \leq \mathcal{U}^\varepsilon(y), \quad y \neq x_0. \end{cases}$$

Then, thanks to the pathwise uniqueness of SDEs (1.2), we have

$$\frac{\mathbf{E}[\varphi(X^\varepsilon(t))] - \varphi(x_0)}{t} \leq -1,$$

By virtue of Dynkin's formula, we have

$$\sum_{i=1}^n b(x_0) \frac{\partial}{\partial x_i} \varphi(x_0) + \frac{\varepsilon^2}{2} \sum_{i,j=1}^n \frac{\partial^2}{\partial x_i \partial x_j} \varphi(x_0) + 1 \leq 0.$$

Similarly, the sub-solution is proved. The proof is complete. \square

Next we will give a stability property of viscosity solutions of (5.5). This property states that if the viscosity solutions V^ε of approximate equations depending on a small parameter ε are uniformly convergent as $\varepsilon \rightarrow 0$, then the limiting function V is a viscosity solution of the limit equation.

Proposition 21. (*Stability result, Lemma 6.2, in [8]*) *Let V^ε be a viscosity subsolution (or super-solution) of*

$$F^\varepsilon(x, V^\varepsilon(x), D_x V^\varepsilon(x), D_x^2 V^\varepsilon(x)) = 0 \quad (5.8)$$

in domain, with some continuous function F^ε satisfying the ellipticity condition. Suppose that F^ε converges to F , uniformly on every compact subset of its domain, and V^ε converges to V , uniformly on compact subset Q of \mathbf{R}^n . Then V is a viscosity subsolution (or a supersolution, respectively) of the limiting equation

$$F(x, V(x), D_x V(x), D_x^2 V(x)) = 0. \quad (5.9)$$

The following theorem will tell us that, for multi-dimensional autonomous cases, under some certain assumptions, we claim that, the limiting solutions "prefer" the optimal solutions of ODE which leave the initial point as fast as possible, that is, the limits solution of perturbed SDE are optimal ones for the exit time of ODE.

Theorem 22. *Assume (H1) and (H2) hold. Given a compact manifold of class $C^{1,1}$, \mathcal{K} such that $\mathcal{V}_\mathcal{K}$ is continuous at 0 and (H3) holds, then we have*

$$\mathbf{E} \left[\tau_\mathcal{K} \left(\tilde{X}^0(\cdot) \right) \right] = \mathcal{V}_\mathcal{K}(0) = \tau_\mathcal{K} \left(\bar{\xi}^0(\cdot) \right), \quad (5.10)$$

where $\tilde{X}^0(\cdot)$ is the limiting solution of the following SDE:

$$\begin{cases} dX^{0,\varepsilon}(t) = b(X^{0,\varepsilon}(t)) dt + \varepsilon dW_t \\ X^{0,\varepsilon}(0) = 0, \quad t \geq 0. \end{cases}$$

Proof. In Section 4, we have shown that $\mathcal{V}_{\mathcal{K}}$ is the unique continuous viscosity solution of H-J-B equation:

$$\begin{cases} -\langle b(x), Du(x) \rangle - 1 = 0, & x \in \mathcal{K}, \\ u(x) = 0, & x \in \partial\mathcal{K}. \end{cases} \quad (5.11)$$

In Section 3, \mathcal{U}^ε is a unique viscosity solution corresponding to the following H-J-B equation:

$$\begin{cases} -\sum_{i=1}^n b(x) \frac{\partial}{\partial X_i} u_n^\varepsilon(x) - \frac{\varepsilon^2}{2} \sum_{i,j=1}^n \frac{\partial^2}{\partial X_i \partial X_j} u^\varepsilon(x) - 1 = 0, & x \in \mathcal{K}, \\ u^\varepsilon(x) = 0, & x \in \partial\mathcal{K}. \end{cases} \quad (5.12)$$

By Proposition 19, Lemma 11, and Lemma 16, we have \mathcal{U}^0 is a unique viscosity solution of (5.11). Hence, we get

$$\mathbf{E}[\theta(x)] = \mathcal{U}^0(x) = \mathcal{V}_{\mathcal{K}}(x), \quad \forall x \in \mathcal{K}.$$

On one hand, clearly, we have

$$X^{x,\varepsilon}(\tau_{\mathcal{K}}(X^{x,\varepsilon}(\cdot))) \in \partial\mathcal{K}, \quad P - \text{a.s.}$$

On the other hand, $\tau_{\mathcal{K}}(X^{x,\varepsilon}(\cdot)) \rightarrow \tilde{t}(\omega)$, as $\varepsilon \rightarrow 0$, that is

$$\tilde{X}^x(\tilde{t}) \in \partial\mathcal{K}, \quad P - \text{a.s.},$$

where $\tilde{X}^x(\cdot)$ denotes the limiting of $X^{x,\varepsilon}(\cdot)$, which implies that

$$\tau_{\mathcal{K}}(\tilde{X}^x(\cdot)) \leq \tilde{t}, \quad P - \text{a.s.}$$

Hence, for almost surely, we have,

$$\liminf_{\varepsilon \rightarrow 0} (\tau_{\mathcal{K}}(X^{x,\varepsilon}(\cdot))) \geq \tau_{\mathcal{K}}(\tilde{X}^x(\cdot)) \geq \mathcal{V}_{\mathcal{K}}(x).$$

By Fatou Lemma, we get

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} \mathbf{E}[\tau_{\mathcal{K}}(X^{x,\varepsilon}(\cdot))] &\geq \mathbf{E}\left[\liminf_{\varepsilon \rightarrow 0} [\tau_{\mathcal{K}}(X^{x,\varepsilon}(\cdot))]\right] \\ &\geq \mathbf{E}\left[\tau_{\mathcal{K}}(\tilde{X}^x(\cdot))\right] \\ &\geq \mathcal{V}_{\mathcal{K}}(x) = \mathbf{E}[\theta(x)]. \end{aligned} \quad (5.5)$$

Therefore, we obtain

$$\mathbf{E}\left[\tau_{\mathcal{K}}(\tilde{X}^x(\cdot))\right] = \mathcal{V}_{\mathcal{K}}(x), \quad x \in \mathcal{K}.$$

Substituting $x = 0$ into (5.5), we get the desired result. \square

We provide a concrete example whose trajectories of optimal solutions are non-symmetric in one-dimensional perturbed SDE. Simultaneously, we give the explicit optimal solutions and validate our major theoretical result (Theorem 20).

Example 23. *Put*

$$b(x) = \begin{cases} x^{\frac{1}{2}}, & x \geq 0, \\ -3|x|^{\frac{1}{2}}, & x < 0. \end{cases} \quad (5.6)$$

Then as $\varepsilon \rightarrow 0$, there are two optimal solutions

$$x_1(t) = \frac{t^2}{4}, \quad x_2(t) = -\frac{9t^2}{4}, \quad t \geq 0.$$

with probability $\frac{1}{1+3^{\frac{2}{3}}}, \frac{3^{\frac{2}{3}}}{1+3^{\frac{2}{3}}}$, respectively (for more information see [4]). Consider the closed interval $\mathcal{K} = [-\frac{9r}{4}, \frac{r}{4}]$, $r > 0$ small enough, while $\psi(x) = \frac{(x+r)^2}{\frac{25r^2}{16}} - 1$. It is fairly to check that $\mathcal{V}_{\mathcal{K}}$ is continuous at 0 by Arzela-Ascoli Theorem. Moreover, $\mathcal{V}_{\mathcal{K}}(0) = \sqrt{r}$, while $\mathbf{E} \left[\tau_{\mathcal{K}} \left(\tilde{X}^0(\cdot) \right) \right] = \sqrt{r} \times \frac{1}{1+3^{\frac{2}{3}}} + \sqrt{r} \times \frac{3^{\frac{2}{3}}}{1+3^{\frac{2}{3}}} = \sqrt{r}$, where $\tilde{X}^0(\cdot)$ denotes the limiting of perturbed SDE with drift b defined in (5.6). Hence $\mathcal{V}_{\mathcal{K}}(0) = \mathbf{E} \left[\tau_{\mathcal{K}} \left(\tilde{X}^0(\cdot) \right) \right]$ holds.

The following two examples show that the dimension of the Brownian motion with respect to the dimension of the state plays an important role.

Example 24. *Consider the following two-dimensional SDE:*

$$\begin{cases} dX^\varepsilon(t) = 2\text{sign}(X^\varepsilon(t)) \sqrt{|X^\varepsilon(t)|} dt + \varepsilon dW_t^1, \\ dY^\varepsilon(t) = 2\text{sign}(Y^\varepsilon(t)) \sqrt{|Y^\varepsilon(t)|} dt + \varepsilon dW_t^2, \\ (X(0), Y(0)) = (0, 0), \end{cases} \quad (5.7)$$

where $W^i, i = 1, 2$ are two independent Brownian motions. From Example 1, we have

$$\begin{cases} X_1^0(t) = (t^2, t^2) \\ X_2^0(t) = (t^2, -t^2) \\ X_3^0(t) = (-t^2, t^2) \\ X_4^0(t) = (-t^2, -t^2) \end{cases}$$

with probability $\frac{1}{4}$, respectively, furthermore, the exit time is $\frac{\sqrt{2r}}{2}$ from the ball $B(0, r)$ for some small $r > 0$.

Remark 25. *Consider the example similar to (5.7) but with a one dimensional Brownian motion $(W_t)_{t \geq 0}$,*

$$\begin{cases} dX^\varepsilon(t) = 2\text{sgn}(X^\varepsilon(t)) \sqrt{|X^\varepsilon(t)|} dt + \varepsilon dW_t, \\ dY^\varepsilon(t) = 2\text{sgn}(Y^\varepsilon(t)) \sqrt{|Y^\varepsilon(t)|} dt + \varepsilon dW_t, \\ (X(0), Y(0)) = (0, 0). \end{cases}$$

Clearly, $X(\cdot)$ and $Y(\cdot)$ have the same finite-dimensional distributions on the same probability and state space. Hence we claim that

$$\begin{cases} X^0(t) = (t^2, t^2) \\ Y^0(t) = (-t^2, -t^2) \end{cases}$$

are limit solutions with probability $\frac{1}{2}$, respectively.

We end this section with a two-dimensional couple case which shows infinite many optimal solutions.

Example 26. Consider two-dimensional SDE, W_t^1 and W_t^2 are two independent Brownian motions,

$$\begin{cases} dX_1(t) = \frac{2X_1(t)}{\left[\left(\frac{X_1}{a}\right)^2 + \left(\frac{X_2}{b}\right)^2\right]^{\frac{1}{4}}} dt + \varepsilon dW_t^1 \\ dX_2(t) = \frac{2X_2(t)}{\left[\left(\frac{X_1}{a}\right)^2 + \left(\frac{X_2}{b}\right)^2\right]^{\frac{1}{4}}} dt + \varepsilon dW_t^2, \\ (X_1(0), X_2(0)) = (0, 0), \end{cases}$$

where $a > b > 0$.

We study the following ODE

$$\begin{cases} dx_1(t) = \frac{2x_1(t)}{\left[\left(\frac{x_1}{a}\right)^2 + \left(\frac{x_2}{b}\right)^2\right]^{\frac{1}{4}}} dt \\ dx_2(t) = \frac{2x_2(t)}{\left[\left(\frac{x_1}{a}\right)^2 + \left(\frac{x_2}{b}\right)^2\right]^{\frac{1}{4}}} dt \\ (x_1(0), x_2(0)) = (0, 0). \end{cases}$$

The solutions are

$$\begin{cases} x_1(t) = at^2 \cos \theta \\ x_2(t) = bt^2 \sin \theta, \quad \theta \in [0, 2\pi]. \end{cases}$$

Then, under some elliptic test domain, for example, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, it is easy to verify that all the rays starting from origin are optimal solutions.

6 Conclusion

In this article, we have shown that limit solutions of perturbed SDE are optimal solutions for the exit time of multi-dimensional ODE. The theory developed in this paper extends one-dimensional case to multi case.

Acknowledgement. The author thanks Prof. Marc Quincampoix for introducing the subject and for conversations, ideas and advice. The author is also indebted to Prof. Andrzej Swiech, Prof. Franco Flandoli, Prof. Ying Hu, Prof. Rainer Buckdahn, Prof. Khaled Bahlali, and Prof. François Delarue for their helpful conversations, especially thanks to Prof. Andrzej Swiech and Prof. Franco Flandoli to direct new idea to deal with the second order H-J-B equations.

References

- [1] J.-P. Aubin, *Viability Theory*, Birkhauser, 1992.
- [2] J.-P. Aubin, H. Frankowska, *Set-valued analysis, Systems and control: Foundations and Applications*, vol. 2, Birkhauser Boston Inc., Boston, MA, 1990, ISBN 0-8176-3478-9. MR 1048347.
- [3] R. Bafico, and P. Baldi, (1982), Small random perturbation of Phenomena, *Stochastic* 6 279-292.
- [4] R. Bafico, (1980), On the convergence of the weak solutions of stochastic differential equations when the noise intensity goes to zero. *Boll. Unione Mat. Ital. Sez. B* 17, 308-324.
- [5] M. Bardi and I. Capuzzo-Dolcetta, "Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations," *Systems and Control: Foundations and Applications*, Birkhäuser, Boston, 1997.
- [6] R. Buckdahn, Y. Ouknine, and M. Quincampoix, (2009), On limiting values of stochastic differential equations with small noise intensity tending to zero, *Bull. Sci. Math* 133: 229-237.
- [7] M. I. Friedlin, A. D. Wentzell, *Random perturbations of dynamical systems*, *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, vol. 260, Springer-Verlag, New York, 1984.
- [8] Wendell H. Fleming, H. Mete Soner, *Controlled Markov Processes and Viscosity Solutions*, Springer, second edition.
- [9] F. Flandoi, *Random Perturbation of PDEs and Fluid Dynamic Models*, Springer, 2011.
- [10] D. Gilbarg, N. Trudinger, *Elliptic Partial Differential Equations of Second Order*. Springer.
- [11] M. Gradinaru, S. Herrmann, B. Roynette, (2001), A singular large deviations phenomenon, *Ann. Inst. H. Poincaré Probab. Statist.* 37 (5) 555-580.
- [12] G. Gabor, M. Quincampoix, (2002) On Existence of solutions to differential equations or inclusions remaining in a prescribed closed subset of a finite-dimensional space, *Journal of Differential Equations* 185, 483-512.
- [13] I. Gyöngy, T. Martinez, (2001), On stochastic differential equations with locally unbounded drift. *Czechoslovak Math. J.* (4) 51 (126) 763-783.
- [14] A.K. Zvonkin, (1974), A transformation of the phase space of a diffusion process that removes the drift, *Mat. Sb.* (1) 93 (135).

- [15] X. Zhang, (2005), Strong solutions of SDES with singular drift and Sobolev diffusion coefficients. *Stochastic Processes and their Applications* 115 1805-1818